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Asia Pacific Center for Theoretical Physics

Excercises

◎ Sign convension

① Lorenzian signature : $(-, +, +, +)$

② Einstein sign : $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = (+) 8\pi G T_{\mu\nu}$

③ Riemann sign : $(+) R^{\lambda}{}_{\mu\nu\kappa} = \partial_\nu \Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\eta\nu} \Gamma^{\eta}_{\mu\kappa} - \Gamma^{\lambda}_{\eta\kappa}$

1. Einstein equation from the action

$$S = \int_M d^4x \sqrt{g} \Big[\underbrace{\frac{1}{16\pi G} (R - 2\Lambda)}_{\text{EH term}} + \mathcal{L}_M \Big] + \oint_{\underset{\text{timelike}}{\partial M}} d^3x \sqrt{h} \, \frac{K - K_0}{8\pi G}$$

K = the trace of the extrinsic curvature of the boundary

K_0 = same object calculated in the usual flat metric

$h = \begin{cases} +\det h_{ab} & \text{(spacelike hypersurface)} \\ -\det h_{ab} & \text{(timelike hypersurface)} \end{cases}$, $g = -\det g_{\mu\nu}$

n_μ : unit vector outward pointing to the boundary

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The extrinsic curvature tensor K_{ab} is a tensor on ∂M of rank $\{^0_2\}$.

$$K_{ab} = n_{\mu;\nu}\, e^{\mu}{}_{a}\, e^{\nu}{}_{b} = n^{\mu} g_{\mu\nu} \Gamma^{\nu}_{ab} = n_{\mu} \Gamma^{\mu}_{ab}$$

$$K_{ab} = K_{ba}, \qquad K = h^{ab} K_{ab} = n^{\mu}{}_{;\mu}$$

$$^{(4)}R^{\lambda}{}_{\mu\nu\kappa}\, h^{a}{}_{\lambda}\, h^{\mu}{}_{b}\, h^{\nu}{}_{m}\, h^{\kappa}{}_{n} = \underbrace{^{(3)}R^{a}{}_{bmn}}_{\text{intrinsic curvature tensor}} - \epsilon \left(K^{a}{}_{m} K_{bn} - K^{a}{}_{n} K_{bm} \right)$$

$$n^{\mu} n_{\mu} = \epsilon = \begin{cases} +1 & (\text{timelike hypersurface}) \\ -1 & (\text{spacelike hypersurface}) \end{cases}$$

$$g_{\alpha\beta} = h_{\alpha\beta} + \epsilon\, n_{\alpha} n_{\beta} \qquad \left(h_{ab} = g_{\alpha\beta}\, e^{\alpha}{}_{a}\, e^{\beta}{}_{b} \right)$$

$$\delta S = \underbrace{\delta S_{EH}}_{①} + \underbrace{\delta S_M}_{②} + \underbrace{\delta S_B}_{③} = 0$$

① $\delta(\sqrt{g}R) = R\,\delta(\sqrt{g}) + \sqrt{g}\,\delta(g^{\mu\nu}R_{\mu\nu})$

$$= -\frac{1}{2}\sqrt{g}\,g_{\mu\nu}R\,\delta g^{\mu\nu} + \sqrt{g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$$

$$= \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$$

$$\delta(g) = \delta(-\det g_{\mu\nu}) = -\exp[\mathrm{Tr}(\log g_{\mu\nu})]\,\delta[\mathrm{Tr}(\log g_{\mu\nu})]$$

$$= (-\det g_{\mu\nu})\,\mathrm{Tr}(g^{-1}_{\mu\nu}\delta g_{\mu\nu}) = g\,g^{\mu\nu}\delta g_{\mu\nu}$$

$$= -g\,g_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}\,g_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta(\sqrt{g}\Lambda) = -\frac{1}{2}\sqrt{g}\,g_{\mu\nu}\Lambda\,\delta g^{\mu\nu}$$

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$ (Palatini identity)

$$g^{\mu\nu}\delta R_{\mu\nu} \equiv \delta V^{\lambda}{}_{;\lambda}, \quad \delta V^{\lambda} \equiv g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu}$$

$$\int_M \sqrt{g}\,d^4x\,g^{\mu\nu}\delta R_{\mu\nu} = \int_M \sqrt{g}\,d^4x\,\delta V^{\lambda}{}_{;\lambda}$$

$$= \oint_{\partial M} d\Sigma_\lambda\,\delta V^\lambda$$ (stoke's Theorem)

$$= \oint_{\partial M} \epsilon\,\delta V^\lambda n_\lambda \sqrt{h}\,d^3x$$

$$\delta\Gamma^{\lambda}_{\mu\nu}\Big|_{\partial M} = \frac{1}{2} g^{\lambda\sigma}\left(\delta g_{\sigma\mu,\nu} + \delta g_{\sigma\nu,\mu} - \delta g_{\mu\nu,\sigma}\right)$$

$$n^{\lambda}\delta v_{\lambda} = n^{\lambda} g^{\mu\nu}\left(\delta g_{\lambda\nu,\mu} - \delta g_{\mu\nu,\lambda}\right)$$

$$n^{\lambda}\delta v_{\lambda}\Big|_{\partial M} = n^{\lambda}\big(h^{\mu\nu} + \epsilon \underbrace{n^{\mu}n^{\nu}}_{\text{symm } (\mu\leftrightarrow\lambda)}\big)\underbrace{\left(\delta g_{\lambda\nu,\mu} - \delta g_{\mu\nu,\lambda}\right)}_{\text{anti-symm } (\mu\leftrightarrow\lambda)}$$

$$= n^{\lambda} h^{\mu\nu}\left(\delta g_{\lambda\nu,\mu} - \delta g_{\mu\nu,\lambda}\right)$$

$$= n^{\lambda} h^{\mu\nu}\left(\delta g_{\lambda\nu;\mu} - \delta g_{\mu\nu;\lambda} + 2\Gamma^{\sigma}_{\mu\nu}\delta g_{\sigma\lambda}\right)$$

$$= -h^{\mu\nu}\delta g_{\mu\nu;\lambda} n^{\lambda}$$

$$\begin{cases} \delta g_{\sigma\lambda} = 0 \quad \text{on } \partial M \\ h^{\mu\nu}\delta g_{\lambda\nu;\mu} = 0 \quad (\text{tangential derivatives of } \delta g_{\lambda\nu} \text{ must vanish!}) \end{cases}$$

⊛ This is nonzero because the normal derivative of $\delta g_{\mu\nu}$ is not required to vanish on ∂M.

The EH term, R, contains second derivatives of the metric tensor. This is a nontypical feature of field theories.

$$② \quad \delta(\sqrt{g}\,\mathcal{L}_M) = \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \mathcal{L}_M \delta\sqrt{g}$$

$$= \left(\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_M g_{\mu\nu}\right)\delta g^{\mu\nu}$$

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + \mathcal{L}_M g_{\mu\nu}$$

$$\delta S_M = -\frac{1}{2}\int_M \sqrt{g}\, d^4x\, T_{\mu\nu}\delta g^{\mu\nu}$$

$$\mathcal{L}_M = -\frac{1}{2} g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)$$

$$\delta(\sqrt{g}\, g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi) = \delta(\sqrt{g})\, g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \sqrt{g}\,\nabla_\alpha\phi\nabla_\beta\phi\,\delta g^{\alpha\beta}$$

$$+ \sqrt{g}\, g^{\alpha\beta}\delta(\nabla_\alpha\phi\nabla_\beta\phi)$$

$$= -\frac{1}{2} g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\sqrt{g}\, g_{\mu\nu}\delta g^{\mu\nu} + \sqrt{g}\,\nabla_\mu\phi\nabla_\nu\phi\,\delta g^{\mu\nu}$$

$$+ 2\sqrt{g}\, g^{\alpha\beta}\left[\nabla_\alpha(\delta\phi)\nabla_\beta\phi\right]$$

$$= \sqrt{g}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\, g_{\mu\nu}\right)\delta g^{\mu\nu} - 2\sqrt{g}\,\phi^{;\alpha}{}_{;\alpha}\delta\phi$$

$$\int\sqrt{g}\, d^4x\, g^{\alpha\beta}\, 2\left[\nabla_\alpha(\delta\phi)\nabla_\beta\phi\right] = 2\int\sqrt{g}\, d^4x\, g^{\alpha\beta}\{\partial_\alpha[(\delta\phi)\nabla_\beta\phi] - (\delta\phi)\partial_\alpha\nabla_\beta\phi\}$$

$$= 2\int\sqrt{g}\, d^4x\, g^{\alpha\beta}\{\underline{\nabla_\alpha[(\delta\phi)\nabla_\beta\phi]} - (\delta\phi)\nabla_\alpha\nabla_\beta\phi\}$$

$$\frac{1}{0}$$

$$= -2\int \sqrt{g}\, d^4x\, g^{\alpha\beta} (\nabla_\alpha \nabla_\beta \phi)\delta\phi$$

$$\delta(\sqrt{g}\,U) = -\frac{1}{2}\sqrt{g}\, g_{\mu\nu}\, U \delta g^{\mu\nu} + \sqrt{g}\,\frac{dU}{d\phi}\delta\phi$$

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③ $$k = n^\mu{}_{;\mu} = \underbrace{(\varepsilon n^\mu n^\nu + h^{\mu\nu})}_{g^{\mu\nu}} n_{\mu;\nu} = h^{\mu\nu} n_{\mu;\nu} = h^{\mu\nu}(n_{\mu,\nu} - \Gamma^\sigma_{\mu\nu}$$

$$\delta k = -h^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} n_\sigma = -\frac{1}{2}h^{\mu\nu}(\delta g_{\lambda\mu,\nu} + \delta g_{\lambda\nu,\mu} - \delta g_{\mu\nu,\lambda})n^\lambda$$

$$= \frac{1}{2}h^{\mu\nu}\delta g_{\mu\nu;\lambda} n^\lambda \quad (\nabla_\lambda \delta g_{\mu\nu} + \Gamma^\sigma_{\lambda\mu}\delta g_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu}\delta g_{\sigma\mu})$$

$$\delta S_B = \oint_{\partial M}\sqrt{h}\, d^3x \frac{\delta k}{8\pi G} = \frac{1}{16\pi G}\oint_{\partial M}\sqrt{h}\, d^3x (h^{\mu\nu}\delta g_{\mu\nu;\lambda} n^\lambda)$$

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④ the term with respect to $\delta g^{\mu\nu}$

$$\Big[\frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}) - \frac{1}{2}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\, g_{\mu\nu})$$
$$+ \frac{1}{2}g_{\mu\nu}U(\phi)\Big]\delta g^{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\, g_{\mu\nu} - g_{\mu\nu}U(\phi)$$

$$R = 4\Lambda + 8\pi G(4U + \nabla^\mu\phi\nabla_\mu\phi)$$

the term with respect to $\delta\phi$

$$\left[\phi^{;\alpha}_{\ ;\alpha} - \frac{\partial U}{\partial \phi} \right] \delta\phi$$

$$\Box\phi = \frac{\partial U}{\partial \phi} \quad \text{or} \quad \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, g^{\mu\nu} \partial_\nu \phi \right) = \frac{\partial U}{\partial \phi}$$

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2. $\pm w^2 + x^2 + y^2 + z^2 = \pm a^2 \ (a^2 > 0)$

$$d\ell^2 = a^2 \left(\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right)$$

curvature index $k = 0, \pm 1$

① ($w = a\cos\chi$, $x = a\sin\chi\cos\theta$, $y = a\sin\chi\sin\theta\cos\phi$, $z = a\sin\chi\sin\theta\sin\phi$)

$$ds^2 = a^2 (d\chi^2 + \sin^2\chi\, d\Omega_2^2)$$

$$(r = \sin\chi, \quad dr^2 = (1-r^2) d\chi^2)$$

$$= a^2 \left(\frac{dr^2}{1-r^2} + r^2 d\Omega_2^2 \right)$$

② ($w = \text{const}$, $\tilde{x} = \frac{x}{a}$, $\tilde{y} = \frac{y}{a}$, $\tilde{z} = \frac{z}{a}$)

$dx = a\, d\tilde{x}$, ..

$$d\ell^2 = a^2 (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) = a^2 (dr^2 + r^2 d\Omega_2^2)$$

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③ $-w^2 + x^2 + y^2 + z^2 = -a^2$, $dl^2 = -dw^2 + dx^2 + dy^2 + dz^2$

$$\left(\begin{array}{l} w = a\cosh\chi, \quad x = a\sinh\chi\sin\theta\cos\phi, \quad y = a\sinh\chi\sin\theta\sin\phi \\ z = a\sinh\chi\cos\theta \end{array} \right)$$

$$dl^2 = a^2 (d\chi^2 + \sinh^2\chi \, d\Omega_2^2)$$

$$(r = \sinh\chi, \quad dr^2 = (1+r^2)d\chi^2)$$

$$= a^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_2^2 \right)$$

$$r = \frac{\bar{r}}{1 + k\bar{r}^2/4}, \qquad dr = \frac{d\bar{r}\left(1 - \frac{k\bar{r}^2}{4}\right)}{\left(1 + \frac{k\bar{r}^2}{4}\right)^2}$$

$$dl^2 = a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right)$$

$$= a^2 \frac{(d\bar{r}^2 + \bar{r}^2 d\Omega_2^2)}{(1 + k\bar{r}^2/4)^2} = a^2 \frac{(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)}{(1 + k\bar{r}^2/4)^2}$$

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3. $ds^2 = -dt^2 + a^2(t)\left(\dfrac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$

$g_{00} = -1, \quad g_{it} = 0, \quad g_{ij} = a^2(t)\tilde{g}_{ij}, \quad g^{00} = -1, \quad g^{ij} = \dfrac{1}{a^2 \tilde{g}_{ij}}$

$\tilde{g}_{ij}$: the metric for a three-dimensional maximally symmetric space

$$\tilde{g}_{rr} = (1-kr^2)^{-1}, \quad \tilde{g}_{\theta\theta} = r^2, \quad \tilde{g}_{\phi\phi} = r^2 \sin^2\theta$$

nonvanishing elements of the affine connection

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\sigma}\left[\frac{\partial g_{\sigma j}}{\partial x^i} + \frac{\partial g_{\sigma i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\sigma}\right] = \frac{g^{00}}{2}\left[\frac{\partial g_{0j}}{\partial x^i} + \frac{\partial g_{0i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^0}\right]$$

$$= a\dot{a}\,\tilde{g}_{ij}$$

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\sigma}\left[\frac{\partial g_{\sigma j}}{\partial x^0} + \frac{\partial g_{\sigma 0}}{\partial x^j} - \frac{\partial g_{0j}}{\partial x^\sigma}\right] = \frac{1}{2} g^{ii}\frac{\partial g_{ij}}{\partial x^0} = \frac{\dot{a}}{a}\delta^i_j$$

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma}\left[\frac{\partial g_{\sigma k}}{\partial x^j} + \frac{\partial g_{\sigma j}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\sigma}\right] = \frac{g^{il}}{2}\left[\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right]$$

$$= \frac{1}{2}(\tilde{g}^{-1})^{il}\left[\frac{\partial \tilde{g}_{lj}}{\partial x^k} + \frac{\partial \tilde{g}_{lk}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right] \equiv \tilde{\Gamma}^i_{jk}$$

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Ricci tensor

$$R_{00} = -\frac{\partial \Gamma^{\lambda}_{0\lambda}}{\partial x^0} - \Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{0\eta} = -\frac{\partial \Gamma^{i}_{0i}}{\partial x^0} - \Gamma^{j}_{0i}\Gamma^{i}_{0j}$$

$$= -3\frac{\ddot{a}}{a}$$

$$R_{ij} = \frac{\partial \Gamma^{\lambda}_{ij}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{i\lambda}}{\partial x^{j}} + \Gamma^{\eta}_{ij}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{i\lambda}\Gamma^{\lambda}_{j\eta}$$

$$= \frac{\partial \tilde{\Gamma}^{k}_{ij}}{\partial x^{k}} - \frac{\partial \tilde{\Gamma}^{k}_{ik}}{\partial x^{j}} + \tilde{\Gamma}^{k}_{ij}\tilde{\Gamma}^{l}_{lk} - \tilde{\Gamma}^{k}_{il}\tilde{\Gamma}^{l}_{jk}$$

$$+ \frac{\partial \Gamma^{0}_{ij}}{\partial x^{0}} - \Gamma^{0}_{il}\Gamma^{l}_{j0} - \Gamma^{k}_{i0}\Gamma^{0}_{jk} + \Gamma^{0}_{ij}\Gamma^{k}_{k0}$$

$$= \tilde{R}_{ij} + (a\ddot{a} + 2\dot{a}^2)\,\tilde{g}_{ij}$$

$$= (a\ddot{a} + 2\dot{a}^2 + 2k)\,\tilde{g}_{ij}$$

space of constant curvature

$$\tilde{R}_{ij} = (n-1)k\,\tilde{g}_{ij}$$

$$\tilde{R}_{ijkl} = k\,(g_{lj}\,g_{ik} - g_{kj}\,g_{il})$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{rr} = \frac{1}{1-kr^2}(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k), \quad R_{\phi\phi} = r^2\sin^2\theta\,(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

(0,0) - component of the Einstein equations

$$R_{00} = 8\pi G \left[T_{00} - \frac{1}{2} g_{00} T^{\lambda}_{\lambda} \right] = -\frac{3\ddot{a}}{a} \quad \cdots (i)$$

$$\Rightarrow g^{00} R_{00} = 8\pi G \left[T^{0}_{0} - \frac{1}{2} T^{\lambda}_{\lambda} \right] = 3\frac{\ddot{a}}{a}$$

(r, r) - component

$$R_{rr} = 8\pi G \left[T_{rr} - \frac{1}{2} g_{rr} T^{\lambda}_{\lambda} \right] = \frac{1}{a^2} g_{rr} \left[2\dot{a}^2 + \ddot{a}a + 2k \right]$$

$$\Rightarrow g^{rr} R_{rr} = 8\pi G \left[T^{r}_{r} - \frac{1}{2} T^{\lambda}_{\lambda} \right] = \left[2\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right) + \frac{2k}{a^2} \right] \quad \cdots (ii)$$

from (i) and (ii)

$$6\left(\frac{\dot{a}}{a}\right)^2 + 6\left(\frac{k}{a^2}\right) = 8\pi G \left[3T^{r}_{r} - \frac{3}{2} T^{\lambda}_{\lambda} \right] - 8\pi G \left[T^{0}_{0} - \frac{1}{2} T^{\lambda}_{\lambda} \right]$$

$$= 8\pi G \left[3T^{r}_{r} - T^{0}_{0} - T^{\lambda}_{\lambda} \right]$$

$$= 8\pi G \left[2T^{r}_{r} - 2T^{0}_{0} - 2T^{\theta}_{\theta} \right]$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a} = \frac{8\pi G}{3} \left[-T^{0}_{0} + T^{r}_{r} - T^{\theta}_{\theta} \right]$$

$$T^{0}_{0} = -\rho_{den}, \quad T^{r}_{r} = T^{\theta}_{\theta} = T^{\phi}_{\phi} = P$$

(0,0) $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a} = \frac{8\pi G \rho_{den}}{3}$ $\leftarrow$ constraint eq.

$$2\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right) + \frac{2k}{a^2} = 8\pi G \left(\frac{\rho}{2} - \frac{P}{2} \right)$$

(i,i) $\Rightarrow$ $2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G P$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

4. $$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$$

$$\nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 8\pi G \nabla_\nu T^{\mu\nu}$$

In a locally inertial coordinate system

$$\Gamma^{x}_{\mu\nu} = 0 \quad \text{at } x.$$

Bianchi identities

$$g^{\lambda\nu}\left(R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0\right)$$

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0$$

$$g^{\mu\kappa}\left(\qquad \qquad \qquad \right)$$

$$R_{;\eta} - R^{\kappa}{}_{\eta;\kappa} - R^{\nu}{}_{\eta;\nu} = 0$$

$$R_{;\eta} - 2R^{\mu}{}_{\eta;\mu} = 0$$

$$R^{\mu}{}_{\eta;\mu} - \frac{1}{2} R_{;\eta} = \left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0$$

$$g^{\nu\eta}\left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0$$

$$\therefore \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = G^{\mu\nu}{}_{;\mu} = 0$$

$$\Rightarrow \nabla_\nu T^{\mu\nu} = 0.$$

- Bianchi's first identity

$$R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} = 0$$

- Bianchi's second identity

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\nu\beta\gamma;\alpha} + R^{\mu}{}_{\nu\gamma\alpha;\beta} = 0$$

- Symmetry properties

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \qquad , \left(\begin{array}{l} R^{\alpha}{}_{\mu\nu\alpha} = -R^{\alpha}{}_{\mu\alpha\nu} \\ R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \end{array} \right.$$

- In a locally inertial coordinate system ($\Gamma^{\lambda}_{\mu\nu} = 0$ at x)

$$R_{\alpha\beta\gamma\delta;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left[\frac{\partial^2 g_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}} - \frac{\partial^2 g_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}} \right]$$

$$R_{\alpha\beta\delta\eta;\gamma} = \frac{1}{2}\frac{\partial}{\partial x^{\gamma}}\left[\qquad \right]$$

$$R_{\alpha\beta\eta\gamma;\delta} = \frac{1}{2}\frac{\partial}{\partial x^{\delta}}\left[\qquad \right]$$

5. Convert the following quantities $(\hbar = c = k_B = 1)$

① $T_0 = 2.725K$, $k_B = eV/11605K$

$k_B\, 2.725k = \left(\frac{2.725}{11605}\right) eV \simeq 2.348 \times 10^{-4} eV$

② $\rho_\gamma = \frac{\pi^2 T_0^4}{15} = 2.000 \times 10^{-15} eV^4$

$\{(\hbar c)^3 = (1.973 \times 10^{-5} eV\,cm)^3\}$

$= \frac{2.000 \times 10^{-15} eV^4}{(1.973 \times 10^{-5} eV\,cm)^3} = 0.2604\, eV\, cm^{-3}$

$1eV = 1.783 \times 10^{-33} g$

$\Rightarrow \rho_\gamma = 4.643 \times 10^{-34}\, g\, cm^{-3}$

③ $\boxed{\frac{1}{H_0} = 9.3h^{-1} \times 10^{27} cm}$ $H_0 = 100h\, Km\, sec^{-1}\, Mpc^{-1}$, $Mpc = 3.1 \times 10^{19} km$

$H_0 = 3.23h \times 10^{-18} sec^{-1}$ $(C = 3 \times 10^{10} cm/sec)$

$= 1.1h \times 10^{-28} cm^{-1}$

④ $M_p = 1.2 \times 10^{19} GeV = 1.2 \times 10^{28} eV$ $(k_B^{-1} = 11605k/eV)$

$= 1.4 \times 10^{32} k = 6.1 \times 10^{32} cm^{-1}$ $(\hbar c = 1.97 \times 10^{-5} eV\,cm)$

$= 1.8 \times 10^{43} sec^{-1}$

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5 (a) $P = \frac{g}{2\pi^2}\int_m^\infty \frac{(E^2-m^2)^{1/2}E^2}{e^{(E-\mu)/T}\pm 1}dE$, $P = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{e^{(E-\mu)/T}\pm 1}dE$

(i) In Bose-Einstein statistics (relativistic, $T \gg m, \mu$)

$$P_{BE} \simeq \frac{g}{6\pi^2}\int_0^\infty \frac{E^3}{e^{E/T}-1}dE \ , \quad \rho_{BE} \simeq \frac{g}{2\pi^2}\int_0^\infty \frac{E^3}{e^{E/T}-1}dE$$

$$(E/T \equiv x, \quad dE = T\,dx)$$

$$P_{BE} = \frac{g}{6\pi^2}T^4\int_0^\infty \frac{x^3dx}{e^x-1} = \frac{gT^4}{6\pi^2}3!\,\zeta(4) = \frac{g\pi^2}{90}T^4 \ , \quad \rho_{BE} = \frac{g\pi^2}{30}T^4$$

Riemann Zeta function

$$\int_0^\infty dx\frac{x^n}{e^x-1} = n!\,\zeta(n+1)$$

$$\zeta(n) = \sum_{j=1}^\infty \frac{1}{j^n} \ , \quad \zeta(2)=\frac{\pi^2}{6},\ \zeta(3)=\frac{\pi^3}{25},\ \zeta(4)=\frac{\pi^4}{90}\cdots$$

$$\zeta(1) = \infty \ (\text{harmonic series})$$

$$P_{BE} = \frac{1}{3}\rho_{BE}$$

(ii) In Fermi-Dirac statistics

$$P_{FD} \simeq \frac{g}{6\pi^2}\int_0^\infty \frac{E^3}{e^{E/T}+1}dE \ , \quad \rho_{FD} \simeq \frac{g}{2\pi^2}\int_0^\infty \frac{E^3}{e^{E/T}+1}dE$$

$$I^+ \equiv \int_0^\infty \frac{x^3dx}{e^x+1} \ , \quad I^- \equiv \int_0^\infty \frac{x^3dx}{e^x-1}$$

$$I^- - I^+ = \int_0^\infty x^3dx\left(\frac{1}{e^x-1} - \frac{1}{e^x+1}\right) = \int_0^\infty x^3dx\frac{2}{e^{2x}-1}$$

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$$(2x \equiv y \quad , \quad 2dx = dy)$$

$$= \frac{1}{8}\int_0^\infty \frac{y^3\, dy}{e^y - 1} = \frac{1}{8} I^-$$

$$\therefore I^+ = \frac{7}{8} I^-$$

$$P_{FD} = \frac{7}{8} P_{BE}, \quad \rho_{FD} = \frac{7}{8}\rho_{BE} \quad , \quad P_{FD} = \frac{1}{3}\rho_{FD}$$

(b) and (c) in 김항배 교수님 노트

7. sound speed

$$C_s = \sqrt{\frac{dP_t}{d\rho_t}} \qquad P_t = P_\gamma + P_b \;,\; \rho_t = \rho_\gamma + \rho_b$$

$$P_\gamma = \frac{1}{3}\sigma^* T^4 \;,\; \rho_\gamma = \sigma^* T^4$$

$$= \sqrt{\frac{\frac{4}{3}\sigma^* T^3 \delta T}{\delta\rho_b + 4\sigma^* T^3 \delta T}}$$

$$= \sqrt{\frac{1}{3\left(1 + \frac{\delta\rho_b}{4\sigma^* T^3 \delta T}\right)}}$$

for adiabatic perturbations

$$\frac{3\delta T}{T} = \frac{\delta\rho_b}{\rho_b}$$

$$= \sqrt{\frac{1}{3\left(1 + \frac{3\rho_b}{4\sigma^* T^4}\right)}} = \sqrt{\frac{1}{3\left(1 + \frac{3\rho_b}{4\rho_\gamma}\right)}} = \sqrt{\frac{1}{3(1+R)}} \quad \left(\text{pure radiation } C_s = \sqrt{\frac{1}{3}}\right.$$

$$R \equiv \frac{3\rho_b}{4\rho_\gamma} = \frac{3}{4}\frac{\rho_{b0}\left(\frac{a_0}{a}\right)^3}{\rho_{\gamma 0}\left(\frac{a_0}{a}\right)^4} = \frac{3}{4}\frac{\rho_{b0}}{\rho_{\gamma 0}}\left(\frac{a}{a_0}\right) = \frac{3}{4}\frac{\rho_{b0}}{\rho_{\gamma 0}}\frac{1}{(1+z)}$$

$$\left(\rho_{\gamma 0} = 4.5 \times 10^{-34}\, g/cm^3, \quad \rho_{b0} = \Omega_b h^2 \times 1.88 \times 10^{-29}\, g/cm^3\right)$$

$$R \simeq 3.04 \times 10^4 (\Omega_b h^2)\frac{1}{1+z} = 3.04 \times 10^4 (\Omega_b h^2)\frac{T_0}{T}$$

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11. from Friedmann eq.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3}$$

$$1 + \frac{k}{H^2 a^2} = \Omega$$

density parameter or cosmological parameter

$$\Omega \equiv \frac{\rho(t)}{\rho_{cr}(t)} = \frac{8\pi G\rho}{3H^2}, \quad \left(\rho_{cr}(t) = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 = \frac{3H^2}{8\pi G}\right)$$

$$\Omega - 1 = \frac{k}{(Ha)^2} \propto \frac{1}{\rho a^2} \propto \begin{cases} a, & MD \\ a^2, & RD \end{cases}$$

$$t = t_{BBN} \sim 1s \;:\; |\Omega - 1| \lesssim 10^{-16}.$$

$$t = t_{pl.} \sim 10^{-43}s \;:\; |\Omega - 1| \lesssim 10^{-60}$$

$$\Omega - 1 = (\Omega_0 - 1)\frac{(Ha)_0^2}{(Ha)_i^2} = (\Omega_0 - 1)\left(\frac{\dot{a}_0}{\dot{a}_i}\right)^2$$

as a function of temperature, in the radiation dominated era,

$$\Omega(T) - 1 \cong 4 \times 10^{-15}(\Omega_0 - 1)\left(\frac{T}{1MeV}\right)^{-2} \simeq 10^{-60}$$

$$T_{pl} \cong 10^{19}\,GeV$$

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12 $$\frac{1}{\sqrt{g}}\partial_\mu\left[\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi\right]=\frac{dV}{d\phi},\qquad \phi=\phi(t),\quad \sqrt{g}=a^3r^2\sin\theta$$

$$\frac{1}{a^3r^2\sin\theta}\partial_0\left[a^3r^2\sin\theta\,\partial^0\phi\right]=\frac{dV}{d\phi}$$

$$\ddot\phi+3\frac{\dot a}{a}\dot\phi=-\frac{dV}{d\phi}\qquad \left(V=\frac{1}{2}m^2\phi^2\right)$$

$$\ddot\phi+3H\dot\phi=-m^2\phi$$

$$H^2=\frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2+\frac{1}{2}m^2\phi^2\right)\Rightarrow\dot\phi^2+m^2\phi^2=\frac{3}{4\pi G}H^2$$

$$\dot\phi=\sqrt{\frac{3}{4\pi G}}H\sin\theta,\quad m\phi=\sqrt{\frac{3}{4\pi G}}H\cos\theta$$

$$\ddot\phi=-3H\dot\phi-m^2\phi=-3\sqrt{\frac{3}{4\pi G}}H^2\sin\theta-m^2\phi$$

$$=-\sqrt{12\pi G}\,(\dot\phi^2+m^2\phi^2)\sin\theta-m^2\phi$$

$$\ddot\phi=\sqrt{\frac{3}{4\pi G}}\dot H\sin\theta+\sqrt{\frac{3}{4\pi G}}H\cos\theta\,\dot\theta=-\sqrt{12\pi G}\,\dot\phi^2\sin\theta+m\phi\dot\theta$$

$$\left(\begin{array}{l}2H\dot H=\frac{8\pi G}{3}(\dot\phi\ddot\phi+m^2\phi\dot\phi)=\frac{8\pi G}{3}(-3H\dot\phi)=-8\pi GH\dot\phi^2\\ \dot H=-4\pi G\dot\phi^2\end{array}\right)$$

$$\dot\theta=-m-\underbrace{\frac{3}{2}H\sin2\theta}\qquad(m\gg H)$$

This term describes oscillations with decaying amplitude.

$\theta\simeq -mt$

$\Rightarrow$ Scalar field oscillates with frequency $\omega\simeq m$.

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$$\dot{H} = -3H^2 \sin^2 mt \qquad -\frac{dH}{H^2} = 3\sin^2 mt\, dt = 3\,\frac{(1-\cos 2mt)}{2}\, dt$$

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}\left(1 - \frac{\sin 2mt}{2mt}\right)^{-1}$$

$$a \propto t^{2/3}\left(1 - \frac{\cos(2mt)}{t^2} + \cdots\right)$$

$$MP: \quad a \propto t^{2/3} \Rightarrow \rho \propto a^{-3}$$

13. the slow roll parameters ϵ and

$$\boxed{\begin{aligned} \rho &= \tfrac{1}{2}\dot{\phi}^2 + V \\ P &= \tfrac{1}{2}\dot{\phi}^2 - V \end{aligned}}$$

$$\ddot{\phi} \ll 3H\dot{\phi}, \quad V > \dot{\phi}^2$$

$$3H\dot{\phi} \simeq -V', \quad H^2 = \frac{8\pi G}{3} V$$

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi} \Rightarrow \ddot{\phi} = \left(-\frac{\dot{H}}{H^2} - \frac{V''}{3H^2}\right) H\dot{\phi}$$

$$\left(-\frac{\dot{H}}{H^2} = \frac{4\pi G (V')^2}{9H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \quad \frac{V''}{3H^2} = \frac{1}{8\pi G}\left(\frac{V''}{V}\right)\right)$$

$$\ddot{\phi} = \Bigg(\underbrace{\frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2}_{\varepsilon} - \underbrace{\frac{1}{8\pi G}\left(\frac{V''}{V}\right)}_{\delta}\Bigg) H\dot{\phi}$$

$$V = \lambda\phi^4, \quad V' = 4\lambda\phi^3, \quad V'' = 12\lambda\phi^2$$

$$\varepsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{1}{16\pi G}\left(\frac{4}{\phi}\right)^2 = \frac{1}{\pi G\phi^2} \ll 1.$$

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(8)

slow roll condition

$$\delta \simeq \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = \frac{1}{8\pi G}\frac{12}{\phi^2} = \frac{3}{2\pi G\phi^2} \ll 1 \quad \boxed{\phi \gg 1}$$

end of inflation $(\epsilon \simeq 1,\ \delta \simeq 1)$

$$\phi_e \simeq \sqrt{\frac{1}{\pi G}} = \frac{m_{pl}}{\sqrt{\pi}}$$

In the slow roll regime

$$3H\dot{\phi} = -\frac{dV}{d\phi}, \qquad H = \frac{d\ln a}{dt} \simeq \sqrt{\frac{8\pi G V}{3}}$$

(7)

$$\frac{d\ln a}{dt} = \dot{\phi}\frac{d\ln a}{d\phi} \simeq -\frac{dV}{d\phi}\frac{1}{3H}\frac{d\ln a}{d\phi}$$

$$-\frac{dV}{d\phi}\frac{d\ln a}{d\phi} \simeq 8\pi G V$$

$$a(\phi) \simeq a_i \exp\left(8\pi G\int_{\phi}^{\phi_i}\frac{V}{V'}d\phi\right)$$

the number of e-foldings

$$N(t) \equiv \ln\frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} H\,dt \simeq \int_{\phi_{end}}^{\phi}\frac{3H^2}{V'}d\phi \simeq 8\pi G\int_{\phi_{end}}^{\phi}\frac{V}{V'}d\phi$$

$$= \pi G\left[\phi^2\right]_{\frac{m_{pl}}{\sqrt{\pi}}}^{\phi} = \pi G\left(\phi^2 - \frac{1}{\pi G}\right) = \pi G\phi^2 - 1$$

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from
②

$$3\sqrt{\frac{8\pi G\lambda\phi^4}{3}}\,\frac{d\phi}{dt} = -4\lambda\phi^3$$

$$\int \frac{d\phi}{\phi} = -\sqrt{\frac{2\lambda}{3\pi G}}\int dt$$

$$\phi = \phi_i \exp\left[-\sqrt{\frac{2\lambda}{3\pi G}}\,(t-t_i)\right]$$

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